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Nonlinear stochastic dynamics of an array of coupled micromechanical oscillators

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Abstract

The stochastic response of a multi-degree-of-freedom nonlinear dynamical system is determined based on the recently developed Wiener path integral (WPI) technique. The system can be construed as a representative model of electrostatically coupled arrays of micromechanical oscillators, and relates to an experiment performed by Buks and Roukes. Compared to alternative modeling and solution treatments in the literature, the paper exhibits the following novelties. First, typically adopted linear, or higher-order polynomial, approximations of the nonlinear electrostatic forces are circumvented. Second, for the first time, stochastic modeling is employed by considering a random excitation component representing the effect of diverse noise sources on the system dynamics. Third, the resulting high-dimensional, nonlinear system of coupled stochastic differential equations governing the dynamics of the micromechanical array is solved based on the WPI technique for determining the response joint probability density function. Comparisons with pertinent Monte Carlo simulation data demonstrate a quite high degree of accuracy and computational efficiency exhibited by the WPI technique. Further, it is shown that the proposed model can capture, at least in a qualitative manner, the salient aspects of the frequency domain response of the associated experimental setup.

KEYWORDS
Wiener path integral, nonlinear system, stochastic dynamics, nanomechanics

1 | INTRODUCTION

Nanowires are considered to be increasingly important structural building blocks for future nanotechnologies. Indicatively, recent advances in micro- and nano-electro-mechanical devices have enabled fast, reliable, and label-free molecular detection.[1,2] Current and future applications include the detection of chemicals and biological species related to specific diseases. Notably, the detection efficiency depends on various factors such as the presence of stochasticity and nonlinearities. These factors play a key role in understanding the detection principles and eventually optimizing the design of nanomechanical systems and devices.

Specifically, micro/nano-oscillators can exhibit nonlinear/hysteretic response behaviors due to various geometrical configurations and damping mechanisms.[3–5] Further, due to their small sizes, they are subject to various intrinsic sources of stochastic noise such as adsorption–desorption and thermally induced noises.[6,7] Furthermore, current technology enables the fabrication of large arrays, composed of hundreds to tens of thousands of micro/nano-beams, coupled by electric, magnetic, or elastic forces. The benefit from such coupling mechanisms relates to potential enhancement of the nanomechanical system detection sensitivity; see, for instance, the pioneering experimental work by Buks and Roukes.[8]

Overall, it becomes clear that optimization of the design and enhancement of the detection capabilities of nanomechanical systems and devices dictate, first, modeling the micro/nano-oscillator as a nonlinear multi-degree-of-freedom (multi-DOF) dynamical system subject to stochastic excitation. Second, potent uncertainty propagation methodologies are required for solving the governing equations of motion and for determining the system stochastic response. In other words, a high-dimensional system of coupled nonlinear stochastic differential equations needs to be solved accurately and in a computationally efficient manner for determining response statistics. In this regard, the state-of-the-art solution techniques in stochastic engineering dynamics can be broadly divided into two categories: first, those that exhibit a high degree of accuracy, but the associated computational cost becomes prohibitive with an increasing number of stochastic dimensions, and second, those that can readily treat high-dimensional systems, but provide reliable estimates for low-order response statistics only. The interested reader is directed to indicative standard books[9–11] for a broad perspective. Clearly, the development of versatile solution techniques that exhibit both high accuracy and low computational cost is critical for efficient stochastic response analysis of high-dimensional systems of coupled nonlinear micro/nano-oscillators.

One of the promising solution techniques, recently pioneered in the field of engineering mechanics by Kougioumtzoglou and coworkers,[12,13] relates to the concept of the Wiener path integral (WPI).[14,15] According to the WPI technique,[16] the system response joint transition probability density function (PDF) is expressed as a functional integral over the space of all possible paths satisfying the initial and final conditions in time. Next, employing a functional integral series expansion, the contribution only of the first term is typically considered, pertaining to the path with the maximum probability of occurrence. This is referred to in the literature as the most probable path and corresponds to an extremum of the functional integrand. In this regard, the most probable path, which is used for determining the system response joint transition PDF approximately, is computed by solving a functional minimization problem that takes the form of a deterministic boundary value problem.[17] It is remarked that the WPI technique is capable of treating systems exhibiting diverse nonlinear/hysteretic behaviors[18,19] and subjected to non-white and non-Gaussian stochastic excitations.[20] Further, it was shown by Psaros et al.[21] and by Katsidoniotaki et al.[22] that the associated computational cost can be reduced drastically by using sparse representations for the system response PDF in conjunction with compressive sampling concepts and tools.[23]

Remarkably, there are only a few papers in the literature pertaining to stochastic modeling and analysis of nonlinear micro/nano-oscillators. The vast majority of these research efforts relate to low-dimensional (typically single-DOF) systems, for which an analytical or numerical solution treatment is tractable.[24–28] The few papers referring to large arrays of coupled micro/nano-beams modeled as high-dimensional multi-DOF systems rely on significant simplifications and approximations that reduce, unavoidably, the accuracy degree of the stochastic response estimates. Indicatively, a moments equations solution approach was used by Ramakrishnan and Balachandran[29] for determining the stochastic response of an array of microcantilevers under the assumption of weak coupling. Note, however, that the approximations inherent in the standard moments equations solution scheme[9] render it capable of yielding relatively accurate estimates only for the system response first- and second-order statistics (i.e., mean vector and covariance matrix). In this context, it can be argued that developing efficient and robust procedures for designing and optimizing nanomechanical systems requires the determination of the complete joint response PDF, or at least, of lower-dimensional joint PDFs corresponding to selected response coordinates of interest. In this regard, additional information relating, for example, to low-probability events (e.g., failures), and obtained by accurate estimation of the response PDF tails, can be integrated into the optimization methodology, leading to enhanced design. To this aim, the response PDF of a 100-DOF micromechanical system[30] was determined accurately and in a computationally efficient manner by Petromichelakis and Kougioumtzoglou.[31] This was done by developing a variational formulation of the WPI technique with mixed fixed/free boundary conditions that renders the computational cost independent from the total number of stochastic dimensions.

In this paper, attention is directed to an experiment performed by Buks and Roukes[8] pertaining to an array of coupled micro-resonators. In particular, the authors examined the collective behavior of a 67-element array of electrostatically actuated, doubly clamped gold microbeams driven near the principal parametric resonance. In this regard, optical diffraction was used for identifying the modal response of the system under varying parametric excitation. Subsequently, Lifshitz and Cross[32] proposed a system of coupled nonlinear differential equations governing the response of the microbeam array. The model comprised stiffness and damping nonlinearities, whereas the electrostatic forces were approximated as linear. Applying a perturbation solution treatment, it was shown that the model captures the salient response characteristics in an average sense. Further, the model was extended by Zhu et al.[33,34] by considering a nonlinear approximation of the electrostatic forces.

Note that the aforementioned response analyses by Lifshitz and Cross[32] and Zhu et al.[33,34] were purely deterministic. In other words, the impact of stochastic noise sources on the system modeling and analysis was ignored. Herein, a stochastic version of the model is considered for the first time in the literature. This takes the form of a coupled system of nonlinear stochastic differential equations that is solved based on the WPI technique for determining the micromechanical array joint response PDF. The results are compared with pertinent Monte Carlo simulation (MCS) data for demonstrating the accuracy and computational efficiency of the WPI technique. Further, it is shown that the proposed model can capture, at least in a qualitative manner, the salient aspects of the frequency domain response of the associated experimental setup.[8]

2 | MATHEMATICAL FORMULATION

2.1 | Micromechanical system equations of motion

In the ensuing analysis, the focus is on the experimental setup by Buks and Roukes,[8] which can be construed as a representative case of electrostatically coupled micro/nano-resonators. In particular, the experiment pertained to a fabricated array of 67 doubly clamped microbeams that were parametrically excited by applying a time-varying voltage. In the same paper, the authors proposed a linear model to represent the dynamics of the coupled system of microbeams, that is,

$$\ddot{x}_n + \omega_0^2 x_n - QV(t)^2(2x_n - x_{n-1} - x_{n+1}) = 0, \quad n = 1, \ldots, N. \tag{1}$$

In Equation (1), $N = 67$ is the number of DOF, x_n denotes the displacement of the nth DOF, ω_0 represents the natural frequency of each oscillator, and $V(t) = V_{\rm dc} + V_{\rm ac}(t)$ is the applied voltage, with $V_{\rm dc}$ and $V_{\rm ac}(t)$ being the constant dc and time-varying ac components, respectively. Further, $V_{\rm ac}(t) = V_{\rm ac}\cos(\Omega t)$, where $V_{\rm ac}$ is a constant amplitude and $\Omega = 2\omega_0$. Furthermore, Q is a capacitance coefficient that depends on the medium, the surface area of the beams, and the distance between them. As anticipated based on the complex interactions between the beams, experimental data demonstrated the existence of higher-order harmonics in the frequency content of the system response. Obviously, a parametric resonance analysis based on the theoretical linear model of Equation (1) was unable to capture such a nonlinear response behavior.

To address the inadequacy of the linear model of Equation (1) to capture satisfactorily the rich frequency content of the system response, Lifshitz and Cross[32] proposed a model comprising stiffness and damping nonlinearities of the polynomial kind, that is,

$$\begin{aligned}\ddot{x}_n + \omega_0^2 x_n + \epsilon_1 g_1(x) - QV(t)^2(2x_n - x_{n-1} - x_{n+1})\\ + c_0(2\dot{x}_n - \dot{x}_{n-1} - \dot{x}_{n+1}) + \epsilon_2 g_2(x) = 0, \quad n = 1, \ldots, N,\end{aligned} \tag{2}$$

where

$$g_1(x) = x_n^3 \tag{3}$$

and

$$g_2(x) = (\dot{x}_n - \dot{x}_{n-1})(x_n - x_{n-1})^2 - (\dot{x}_{n+1} - \dot{x}_n)(x_{n+1} - x_n)^2. \tag{4}$$

In Equation (2), ϵ_1 and ϵ_2 are parameters controlling the magnitudes of the stiffness and damping nonlinearities, respectively, and c_0 is a linear damping coefficient. Applying a perturbation solution treatment to the nonlinear Equation (2), it was shown[32] that the model can predict, qualitatively, the salient aspects of the system response in the frequency domain. Note, however, that the electrostatic forces were approximated as linear, in a similar manner as in Equation (1). In fact, this simplification was justified[32] by arguing that the effect of the elastic cubic nonlinearities $g_1(x)$ is significantly stronger relative to the electrostatic force nonlinearities, particularly for the case of small beam thickness compared to the gap between adjacent beams.

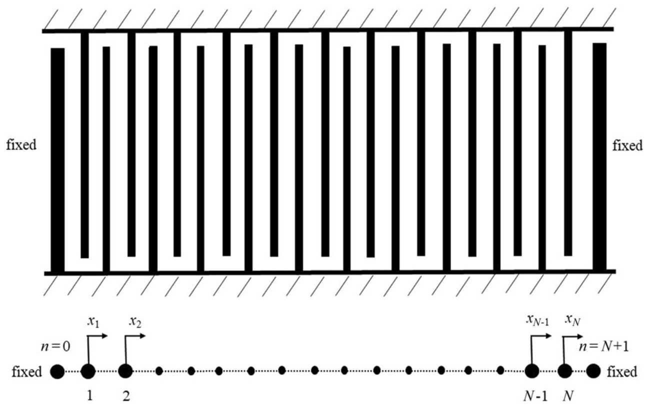

FIGURE 1 An array of coupled micromechanical oscillators and an indicative, approximate model.

To enhance the accuracy of the model of Equation (2) and to also account for more general cases of microbeam arrays, where the gap between adjacent beams is comparable to their thickness (see Figure 1 for an indicative schematic representation), Equation (2) becomes

$$\begin{aligned}\ddot{x}_n + \omega_0^2 x_n + \epsilon_1 g_1(x) - QV(t)^2 g_3(x) + c_0(2\dot{x}_n - \dot{x}_{n-1} - \dot{x}_{n+1})\\ + \epsilon_2 g_2(x) = 0, \quad n = 1, \ldots, N,\end{aligned} \tag{5}$$

where

$$g_3(x) = \frac{1}{(1 + x_{n+1} - x_n)^2} - \frac{1}{(1 + x_n - x_{n-1})^2}. \tag{6}$$

Clearly, the electrostatic forces in Equations (5) and (6) are modeled as nonlinear, exhibiting a decaying behavior with increasing relative distance between the beams. In fact, Zhu et al.[33,34] considered an approximation of Equation (6) by utilizing the first two terms of a Taylor expansion of the nonlinear electrostatic forces. It was shown, based on parametric resonance analysis, that the system response showed a more complex frequency content compared to the model of Equation (2). In passing, note that the linear assumption for the electrostatic forces in the models of Equation (1) and Equation (2) is equivalent to considering the first term only of a Taylor expansion of the nonlinear electrostatic forces.

In all previous models, the impact of stochastic noise sources[6,7] on the system modeling was ignored and the response analyses were purely deterministic. In this paper, a stochastic version of the model of Equation (5) is examined for the first time in the literature by considering a stochastic excitation component. Specifically, Equation (5) is cast in the matrix form

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} + \boldsymbol{g}(\dot{\boldsymbol{x}}, \boldsymbol{x}, t) = \boldsymbol{w}(t), \tag{7}$$

where $\boldsymbol{x} = [x_1, \ldots, x_N]^{\rm T}$, $\boldsymbol{M}, \boldsymbol{C}, \boldsymbol{K}$ denote the $N \times N$ mass, damping, and stiffness matrices, respectively, given by

$$\boldsymbol{M}=\begin{bmatrix}1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & 1\end{bmatrix} \quad (8)$$

$$\boldsymbol{C}=\begin{bmatrix}2c_0 & -c_0 & \cdots & 0\\ -c_0 & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & -c_0\\ 0 & \cdots & -c_0 & 2c_0\end{bmatrix} \quad (9)$$

$$\boldsymbol{K}=\begin{bmatrix}\omega_0^2 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \omega_0^2\end{bmatrix} \quad (10)$$

and

$$\begin{aligned}g(\dot{\boldsymbol{x}},\boldsymbol{x},t)=\Big[&\epsilon_1 x_n^3+\epsilon_2((\dot{x}_n-\dot{x}_{n-1})(x_n-x_{n-1})^2\\&-(\dot{x}_{n+1}-\dot{x}_n)(x_{n+1}-x_n)^2)\\&-QV(t)^2\left(\frac{1}{(1+x_{n+1}-x_n)^2}-\frac{1}{(1+x_n-x_{n-1})^2}\right)\Big]_{N\times 1}\end{aligned} \quad (11)$$

Further, the excitation vector $\boldsymbol{w}(t)$ represents a Gaussian white noise stochastic process with $E[\boldsymbol{w}(t_l)]=E[\boldsymbol{w}(t_{l+1})]=0$ and $E[\boldsymbol{w}(t_l)\boldsymbol{w}^{\mathrm{T}}(t_{l+1})]=\boldsymbol{S}_w\delta(t_{l+1}-t_l)$, where t_l, t_{l+1} are two arbitrary time instants and $\mathbf{S}_w$ denotes an $N\times N$ power spectrum matrix given by

$$\boldsymbol{S}_w=\begin{bmatrix}S_0 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & S_0\end{bmatrix}. \quad (12)$$

2.2 | Micromechanical system stochastic response determination

In this section, the WPI technique developed by Kougioumtzoglou and coworkers[12,13,16–22] for determining the stochastic response of diverse dynamical systems is employed for computing the joint response PDF of the array of coupled microbeams presented in Section 2.1. Specifically, the reduced-order WPI formulation developed by Petromichelakis and Kougioumtzoglou[31] is utilized next for determining any p-dimensional joint response PDF of the system of Equation (7), where $p\le 2N$. This is done in a direct, computationally efficient manner, without computing the $2N$-dimensional joint PDF first and subsequently marginalizing. In the following, the fundamental aspects of the technique are presented for completeness.

It has been shown[15] that the joint response transition PDF $p(\boldsymbol{x}_f,\dot{\boldsymbol{x}}_f,t_f|\boldsymbol{x}_i,\dot{\boldsymbol{x}}_i,t_i)$ can be expressed as a functional integral in the form

$$p(\boldsymbol{x}_f,\dot{\boldsymbol{x}}_f,t_f\mid\boldsymbol{x}_i,\dot{\boldsymbol{x}}_i,t_i)=\int_{\mathcal{C}}\exp\left(-\int_{t_i}^{t_f}\mathcal{L}(\boldsymbol{x},\dot{\boldsymbol{x}},\ddot{\boldsymbol{x}})\mathrm{d}t\right)[\mathrm{d}\boldsymbol{x}(t)], \quad (13)$$

where $\mathcal{C}=\{\boldsymbol{x}_i,\dot{\boldsymbol{x}}_i,t_i;\boldsymbol{x}_f,\dot{\boldsymbol{x}}_f,t_f\}$ is the set of all possible paths with the initial condition $\{\boldsymbol{x}_i,\dot{\boldsymbol{x}}_i,t_i\}$ and the final condition $\{\boldsymbol{x}_f,\dot{\boldsymbol{x}}_f,t_f\}$, $\mathrm{d}\boldsymbol{x}(t)$ denotes a functional measure, and $\mathcal{L}$ represents the Lagrangian functional of the system of Equation (7) given by

$$\begin{aligned}\mathcal{L}(\boldsymbol{x},\dot{\boldsymbol{x}},\ddot{\boldsymbol{x}})=\;&\frac{1}{2}\{\boldsymbol{M}\ddot{\boldsymbol{x}}+\boldsymbol{C}\dot{\boldsymbol{x}}+\boldsymbol{K}\boldsymbol{x}+\boldsymbol{g}(\dot{\boldsymbol{x}},\boldsymbol{x},t)\}^{\mathrm{T}}\boldsymbol{S}_w^{-1}\\&\{\boldsymbol{M}\ddot{\boldsymbol{x}}+\boldsymbol{C}\dot{\boldsymbol{x}}+\boldsymbol{K}\boldsymbol{x}+\boldsymbol{g}(\dot{\boldsymbol{x}},\boldsymbol{x},t)\}.\end{aligned} \quad (14)$$

It is remarked that analytical evaluation of the functional integral of Equation (13) is, in general, an impossible task. Thus, researchers resort, routinely, to the most probable path approximation; that is, only the path $\boldsymbol{x}_c(t)$ is used for the approximate evaluation of Equation (13) for which the stochastic action $\mathcal{S}=\int_{t_i}^{t_f}\mathcal{L}(\boldsymbol{x},\dot{\boldsymbol{x}},\ddot{\boldsymbol{x}})\mathrm{d}t$ is minimized. According to calculus of variations,[35] an extremal of $\mathcal{S}$ can be determined by enforcing the necessary condition that the first variation equals zero, that is, $\delta\mathcal{S}=0$. Next, using a Taylor-type expansion for $\mathcal{S}$ and integrating by parts, the condition $\delta\mathcal{S}=0$ becomes[31]

$$\begin{aligned}&\sum_{n=1}^{N}\left[\left(\mathcal{L}_{\dot{x}_n}-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{\ddot{x}_n}\right)\delta x_n\right]_{t_0}^{t_f}+\sum_{n=1}^{N}[\mathcal{L}_{\ddot{x}_n}\delta\dot{x}_n]_{t_0}^{t_f}\\&+\sum_{n=1}^{N}\int_{t_0}^{t_f}\left(\mathcal{L}_{x_n}-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{\dot{x}_n}+\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{L}_{\ddot{x}_n}\right)\delta x_n\mathrm{d}t=0.\end{aligned} \quad (15)$$

Further, considering fixed initial and final boundary conditions, all variations δx_n and $\delta\dot{x}_n$ equal zero at the boundaries, and thus, the first two terms in Equation (15) vanish. In this regard, Equation (15) leads to the Euler–Lagrange equations

$$\mathcal{L}_{x_n}-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{\dot{x}_n}+\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{L}_{\ddot{x}_n}=0,\ \ n=1,\ldots,N \quad (16)$$

to be solved in conjunction with the $4\times N$ boundary conditions

$$x_n(t_i)=x_{n,i},\ \dot{x}_n(t_i)=\dot{x}_{n,i},\ x_n(t_f)=x_{n,f},\ \dot{x}_n(t_f)=\dot{x}_{n,f},\quad n=1,\ldots,N \quad (17)$$

for computing the most probable path $\boldsymbol{x}_c(t)$. Furthermore, substituting $\boldsymbol{x}_c(t)$ into Equation (13), a specific point of the joint response transition PDF is determined approximately as

$$p(\boldsymbol{x}_f,\dot{\boldsymbol{x}}_f,t_f\mid\boldsymbol{x}_i,\dot{\boldsymbol{x}}_i,t_i)=C\exp\left(-\int_{t_i}^{t_f}\mathcal{L}(\boldsymbol{x}_c,\dot{\boldsymbol{x}}_c,\ddot{\boldsymbol{x}}_c)\mathrm{d}t\right) \quad (18)$$

where C is a normalization constant.

Clearly, in general, the boundary value problem of Equations (16) and (17) is not amenable to an analytical solution treatment, and therefore, numerical schemes are required. In this regard, adopting a brute-force solution approach, for a specific time instant t_f, the values of the joint response PDF are computed based on Equation (18) over a discretized PDF domain of L points in each dimension. This yields L^{2N} boundary value problems to be solved for an N-DOF system governed by Equation (7). Obviously, this leads to an exponential increase in the computational cost with increasing number N of DOF. Eventually, the associated cost becomes prohibitive for large values of N, such as $N=67$ used in the experiment by Buks and Roukes.[8]

To circumvent this limitation, Petromichelakis and Kougioumtzoglou[31] developed a variational formulation with mixed fixed/free boundaries that yields, in a direct manner, a joint response PDF $p(\boldsymbol{u},\boldsymbol{v},t_f\mid\boldsymbol{x}_i,\dot{\boldsymbol{x}}_i,t_i)$ corresponding to a subset only of the components of the response vectors $\boldsymbol{x}_f$ and $\dot{\boldsymbol{x}}_f$, that is,

$\boldsymbol{u} = \{x_{n,f} | n \in U\}$ and $\boldsymbol{v} = \{\dot{x}_{n,f} | n \in V\}$, where U and V are arbitrary subsets $U, V \subseteq \{1, ..., N\}$ with cardinalities $p = |U|$ and $q = |V|$, respectively. Notably, the $(p + q)$-dimensional joint response PDF $p(\boldsymbol{u}, \boldsymbol{v}, t_f | \boldsymbol{x}_i, \dot{\boldsymbol{x}}_i, t_i)$, where $p + q \le 2N$, can be determined at a computational cost that is exponentially related to the dimension $p + q$ of the target PDF only and is independent from the dimension $2N$ of the original system.

In this regard, the path integral representation of Equation (13) becomes

$$p(\boldsymbol{u}, \boldsymbol{v}, t_f \mid \boldsymbol{x}_i, \dot{\boldsymbol{x}}_i, t_i) = \int_C \exp\left(-\int_{t_i}^{t_f} \mathcal{L}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}) \mathrm{d}t\right)[\mathrm{d}\boldsymbol{x}(t)], \tag{19}$$

where $\mathcal{C} = \{\boldsymbol{x}_i, \dot{\boldsymbol{x}}_i, t_i; \boldsymbol{u}, \boldsymbol{v}, t_f\}$ denotes the set of all possible paths with the initial state $\{\boldsymbol{x}_i, \dot{\boldsymbol{x}}_i, t_i\}$ and the final state $\{\boldsymbol{u}, \boldsymbol{v}, t_f\}$. Note that the coordinates $x_{n,f}$ with $n \notin U$ and $\dot{x}_{n,f}$ with $n \notin V$ are considered free. The most probable path, denoted by $\tilde{\boldsymbol{x}}(t)$ in this case, depends on the choice of U and V, since these sets specify which coordinates of $\boldsymbol{x}$ and $\dot{\boldsymbol{x}}$ are fixed at the endpoint ($t = t_f$). According to the fundamental lemma of calculus of variations,[35] the derived Euler–Lagrange Equation (16) is satisfied also by $\tilde{\boldsymbol{x}}$ corresponding to the general class of functions $\boldsymbol{x}(t)$ with arbitrary boundary conditions. Further, for the case of mixed fixed/free boundary conditions considered herein, there are terms in the first two summations of Equation (15) that do not vanish for $n \notin U$ and $n \notin V$. In summary, Equation (15) leads to the Euler–Lagrange Equation (16) in conjunction with the boundary conditions, for $n = 1, ..., N$,

$$\begin{gathered} x_n(t_i) = x_{n,i} \\ \dot{x}_n(t_i) = \dot{x}_{n,i} \\ \begin{Bmatrix} \left[\mathcal{L}_{\dot{x}_n} - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{\ddot{x}_n}\right]_{t=t_f} = 0, & \text{if } n \notin U \\ x_n(t_f) = x_{n,f}, & \text{if } n \in U \end{Bmatrix} \\ \begin{Bmatrix} [\mathcal{L}_{\ddot{x}_i}]_{t=t_f} = 0, & \text{if } n \notin V \\ \dot{x}_n(t_f) = \dot{x}_{n,f}, & \text{if } n \in V \end{Bmatrix} \end{gathered} \tag{20}$$

Finally, solving Equations (16) and (20) numerically yields the most probable path $\tilde{\boldsymbol{x}}(t)$, and a specific point of the (lower-dimensional) joint response PDF is obtained as

$$p(\boldsymbol{u}, \boldsymbol{v}, t_f \mid \boldsymbol{x}_i, \dot{\boldsymbol{x}}_i, t_i) = C\exp\left(-\int_{t_i}^{t_f} \mathcal{L}(\tilde{\boldsymbol{x}}, \dot{\tilde{\boldsymbol{x}}}, \ddot{\tilde{\boldsymbol{x}}}) \mathrm{d}t\right). \tag{21}$$

It is clearly seen that the WPI technique variational formulation with mixed fixed/free boundaries can reduce the associated computational cost drastically by determining directly any lower-dimensional joint response PDF. This capability is particularly advantageous for problems where the interest lies in few $(p + q)$ specific DOF whose stochastic response is critical for the design and optimization of the $2N$-DOF system. In this regard, the L^{2N} boundary value problems required to be solved by the standard formulation of the technique decrease to L^{p+q} problems only, where $30 < L < 50$ is a reasonable range of values for various diverse engineering applications.[16–19] Note that for small values of $p + q$ relative to $2N$, it has been shown[31] that the technique becomes orders of magnitude more efficient than an alternative MCS solution treatment.

3 | NUMERICAL EXAMPLE

In agreement with the experiment carried out by Buks and Roukes,[8] consider a 67-DOF version of the system of Equation (7) with parameter values $\omega_0 = 1$, $\epsilon_1 = 0.1$, $Q = 4 \cdot 10^{-3}$, $V_{ac} = 0.05$, $V_{dc} = 0.5$, $\Omega = 2\omega_0$, $c_0 = 0.01$, $\epsilon_2 = 10^{-3}$, and $S_0 = 10^{-5}$. In the following, the WPI-based PDF estimates are compared with the results obtained by MCS (30 000 realizations). For the MCS, first, excitation realizations compatible with the stochastic process $\boldsymbol{w}(t)$ are generated.[36] Second, a standard Runge–Kutta numerical integration scheme is used for solving Equation (7) and for determining response realizations. Finally, the system response PDF is estimated by performing a statistical analysis on the ensemble of the response realizations. Note that due to the specific type of nonlinearities in Equation (11), and in particular because of the form of the electrostatic forces, it is possible for the effective stiffness of the system to become negative.[37] This is followed, typically, by an unbounded response behavior. Nevertheless, in the numerical example, such an event has practically zero probability of occurrence for the selected parameters. In fact, for relatively small values of the excitation power spectrum magnitude S_0, the restoring force in Equation (7) acts in the direction opposite to the displacement, and the system response exhibits a bounded behavior.

Next, to demonstrate the efficiency and reliability of the WPI technique presented in Section 2.2, only the final displacement x_{10} and velocity $\dot{x}_{10}$ corresponding to the arbitrarily chosen 10th DOF are considered fixed; thus, $p + q = 2$, and the value $L = 31$ is used. This yields 31^2 boundary value problems to be solved for evaluating the joint PDF $p(x_{10}, \dot{x}_{10})$ at a given time instant. The WPI-based joint PDF $p(x_{10}, \dot{x}_{10})$ is plotted in Figure 2A as it evolves with time, and is compared in Figure 2B with MCS-based estimates (30 000 realizations). Obviously, the WPI technique exhibits a quite high degree of accuracy. This is further corroborated in Figure 3, where the WPI-based joint PDF $p(x_{10}, \dot{x}_{10})$ is compared with MCS-based estimates at indicative time instants. In Figure 4, the marginal displacement and velocity PDFs, that is, $p(x_{10})$ and $p(\dot{x}_{10})$, are plotted for time instants $t = 1s$ and $t = 10s$. Comparisons with PDF estimates based on MCS demonstrate an excellent degree of accuracy. Note that in this case, the displacement PDF $p(x_{10})$ at a given time instant (or, similarly, the velocity PDF $p(\dot{x}_{10})$) has been determined by considering only x_{10} fixed at t_f in the WPI technique. From a computational efficiency perspective, this translates to 31 boundary value problems to be solved numerically for evaluating $p(x_{10})$, which required approximately 1 min of computation time, whereas the MCS based on 30 000 realizations required approximately 12 h on the same computer.

Lastly, the capability of the model of Equation (7) to reproduce, at least in a qualitative manner, the frequency domain response of the experimental setup by Buks and Roukes[8] is demonstrated next. Specifically, for a given value of V_{dc}, the frequency content of the response is estimated by applying Fourier transform to an arbitrary

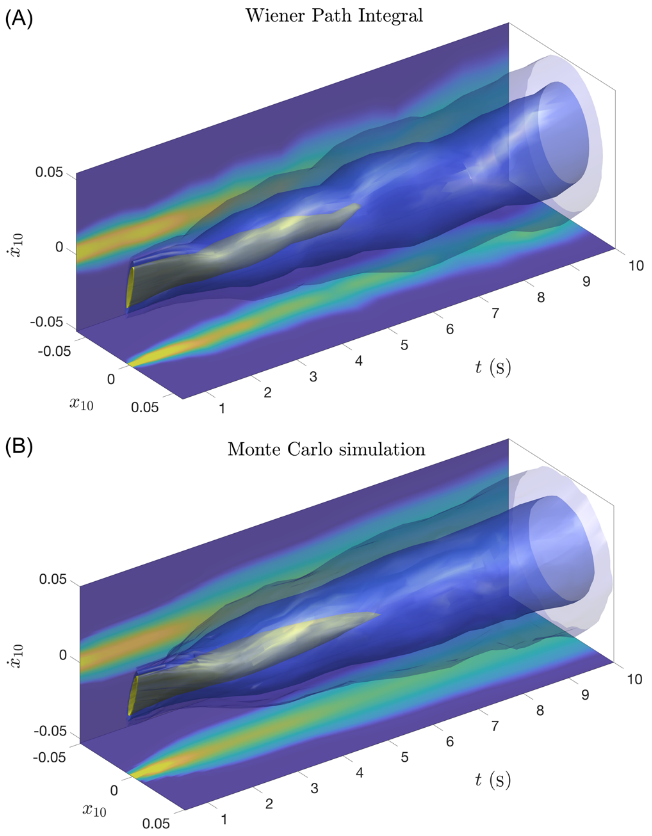

FIGURE 2 Evolution over time of the joint response PDF $p(x_{10}, \dot{x}_{10})$. (A) WPI and (B) MCS estimates (30 000 realizations). The three isosurfaces correspond to PDF values of 10 (light blue), 200 (blue), and 900 (yellow). MCS, Monte Carlo simulation; WPI, Wiener path integral.

response displacement realization corresponding to x_{10}. This is produced by solving the system of Equation (7) excited by a randomly generated white noise realization. The results are plotted in Figure 5, which shows quite similar characteristics to Figure 5 in Buks and Roukes.[8] Clearly, the stochastic nonlinear model of Equation (7) proposed herein is capable of capturing, to a large extent, the salient aspects of the rich frequency content of the system response.

4 | CONCLUDING REMARKS

In this paper, an experimental setup by Buks and Roukes[8] has been considered as a representative case of electrostatically coupled arrays of micro/nano-resonators. Compared to alternative, earlier modeling and solution treatments in the literature,[32–34] the paper exhibits the following novelties: (a) typically adopted linear, or higher-order polynomial, approximations of the nonlinear electrostatic forces have been circumvented; (b) probabilistic modeling has been used for the first time in the literature by considering a stochastic excitation component representing the effect of diverse noise sources on the system dynamics; (c) the resulting high-dimensional, nonlinear system of coupled stochastic differential equations governing the dynamics of the microbeam array has been solved based on a recently developed WPI technique for determining the response joint PDF. Comparisons with pertinent MCS data have demonstrated that the WPI technique exhibits a quite high degree of accuracy and computational efficiency. Further, it has been shown that the

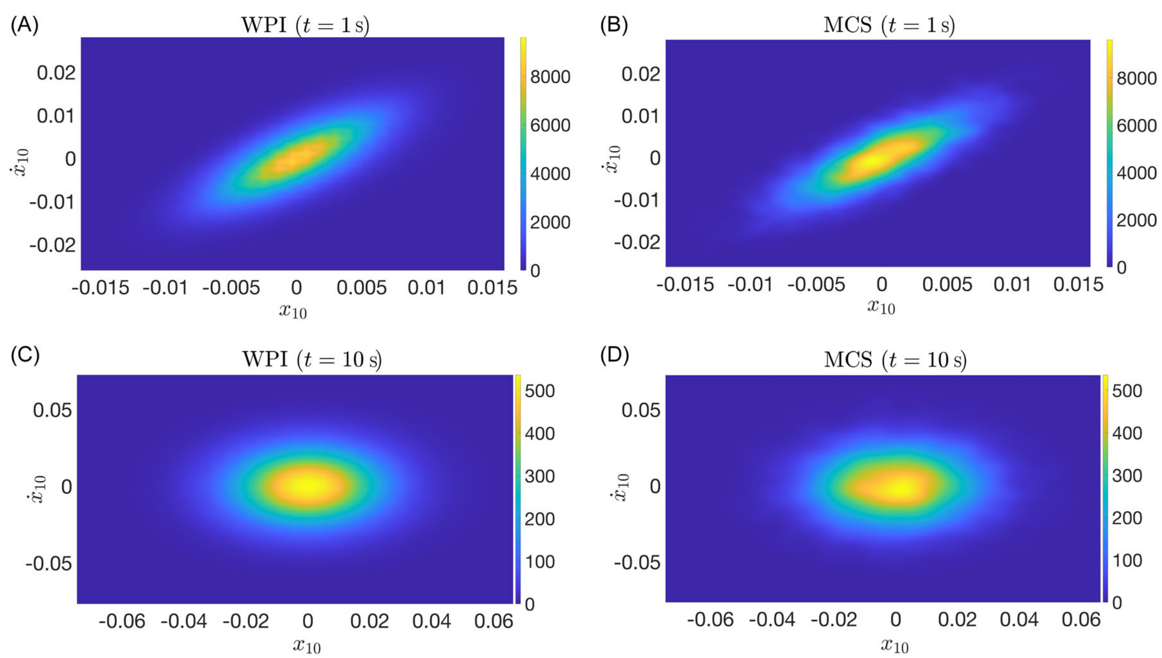

FIGURE 3 Joint response displacement and velocity PDF $p(x_{10}, \dot{x}_{10})$ at two arbitrary time instants and comparisons with MCS data (30 000 realizations): (A) WPI ($t = 1$ s), (B) MCS estimate ($t = 1$ s), (C) WPI ($t = 10$ s), and (D) MCS estimate ($t = 10$ s). MCS, Monte Carlo simulation; WPI, Wiener path integral.

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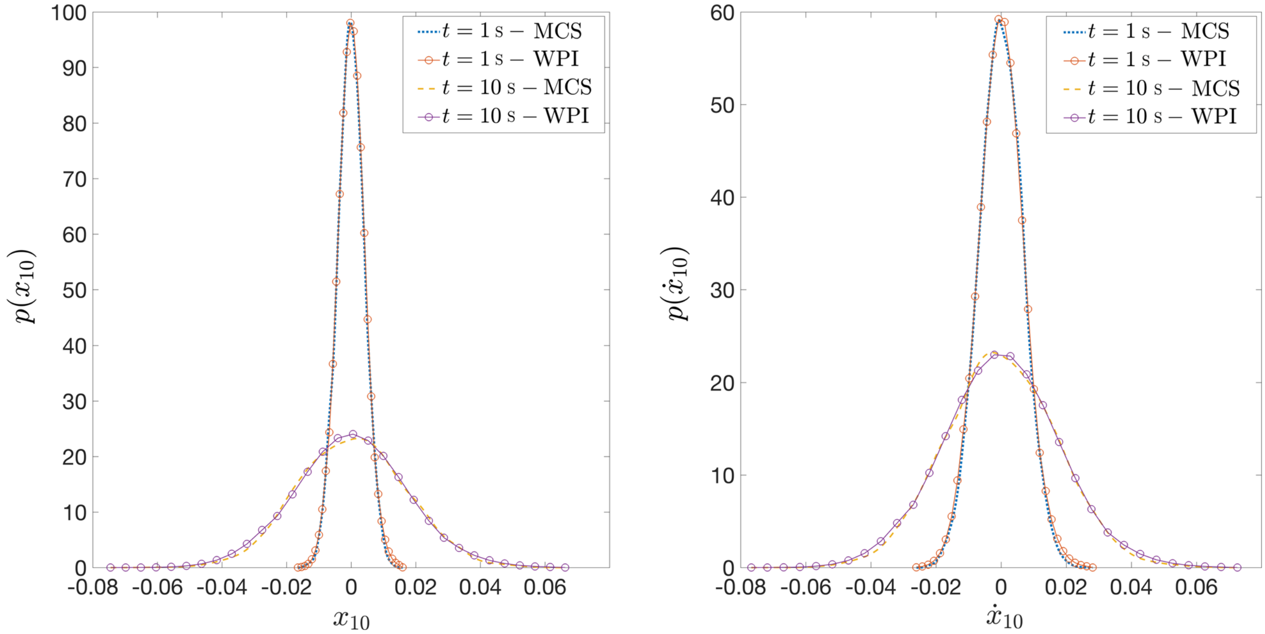

FIGURE 4 Marginal PDFs $p(x_{10})$ and $p(\dot{x}_{10})$ at t = 1 s and t = 10 s obtained by the WPI technique. Comparison with MCS (30 000 realizations). MCS, Monte Carlo simulation; WPI, Wiener path integral.

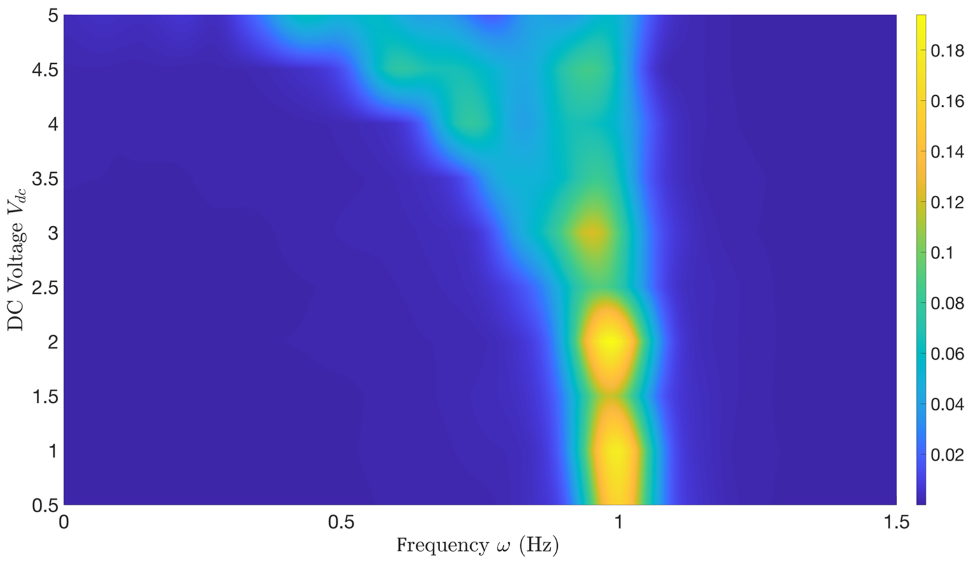

FIGURE 5 Response frequency content estimate as a function of voltage V_{dc}.

proposed model can capture, to a large extent, the salient aspects of the rich frequency content of the system response, and can reproduce, at least in a qualitative manner, the frequency domain response of the experimental setup by Buks and Roukes.[8]

Overall, the WPI technique exhibits, remarkably, both high accuracy and low computational cost. This unique aspect can facilitate the stochastic response analysis of large arrays of micromechanical oscillators to unprecedented levels, thus leading, hopefully, to a paradigm shift in the optimization and design of such systems and devices.

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CONFLICT OF INTEREST STATEMENT

The authors declare no conflicts of interest.

DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available upon reasonable request from the corresponding author.

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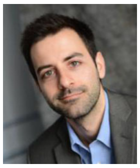

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